2.3 The invertible matrix theorem

Key idea: The invertible matrix theorem provides a fundamental connection between most of the concepts we've studied and the invertibility of a matrix. We can now answer broad questions about both systems of n linear equations in n variables and collections of n vectors from $\mathbb{R}^n$ by only knowing if a matrix is invertible.

The Invertible Matrix Theorem

Let A be a square $n \times n$ matrix. Then the following statements are equivalent. That is, for a given A, the statements are either all true or all false.

a. A is an invertible matrix.
b. A is row equivalent to the $n \times n$ identity matrix.
c. A has n pivot positions.
d. The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
e. The columns of A form a linearly independent set.
f. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.
g. The equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b}$ in $\mathbb{R}^n$.
h. The columns of A span $\mathbb{R}^n$.
i. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$.
j. There is an $n \times n$ matrix C such that $CA = I$.
k. There is an $n \times n$ matrix D such that $AD = I$.
l. A^T is an invertible matrix.

This serves as a broad characterization of invertibility in many contexts.

This result is fundamental: consider it carefully in terms of all the concepts we've studied thus far.

Ex| Determine if $A = \begin{bmatrix} 1 & 0 & -2 \\ 3 & 1 & -2 \\ -5 & -1 & 9 \end{bmatrix}$ is invertible (or any of other properties listed above...)

$$A \sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 4 \\ 0 & -1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 4 \\ 0 & 0 & 3 \end{bmatrix}$$

$\rightsquigarrow$ three pivots! Part (c) of the theorem guarantees A is invertible.

This is an important result that you will become familiar with via your homework. It simply takes practice and time considering it.

To conclude our discussion of invertibility we turn our attention to invertible transforms:

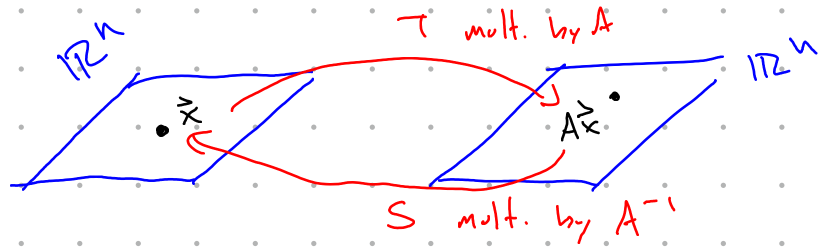
Def: A linear transformation $T: \mathbb{R}^n \to \mathbb{R}^n$ is said to be invertible if there is a function $S: \mathbb{R}^n \to \mathbb{R}^n$ s.t.

$$\text{for all } \vec{x} \text{ in } \mathbb{R}^n, \quad T(S(\vec{x})) = \vec{x} = S(T(\vec{x})).$$

Fact: If S exists it is unique and linear so we may say S is <u>the</u> inverse of T.

T is invertible if and only if it's standard matrix A is invertible (thus the invertible matrix thm implies T is, for example, one-to-one and onto). Furthermore, the standard matrix of S, the inverse of T, is A^{-1} so $S(\vec{x}) = A^{-1}x$.

This fact should be intuitive: $T(S(\vec{x})) = A \cdot A^{-1}\vec{x} = I_n\vec{x} = \vec{x}$ and
$S(T(\vec{x})) = A^{-1} \cdot A\vec{x} = I_n\vec{x} = \vec{x}$.



Ex| If $T: \mathbb{R}^n \to \mathbb{R}^n$ is onto, the columns of it's standard matrix A span $\mathbb{R}^n$. So by the invertible matrix theorem, A is invertible. In particular, item (e) says the columns of A are linearly independent so T <u>must</u> also be one-to-one.

(This argument works in reverse too: T one-to-one $\Rightarrow$ T onto.)